Question 1: Conditional Probability in an Urn Model

Problem Statement

A box contains b white and n black balls. A ball is drawn and replaced with $d+1$ balls of the same color, where d is a positive integer. Compute the probability that the first drawn ball was black, given that the second draw was black.

Solution

Let black and white denote the outcomes of drawing a black ball and a white ball, respectively. We define the probability space as

$$\begin{aligned}
&\Omega = \{\text{black}, \text{white}\}, \\
&\mathcal{F} = \mathcal{P}(\Omega), \\
&\mathbb{P} \text{ is the probability measure on } \mathcal{F}.
\end{aligned}$$

Let A_{black} and A_{white} be the events that a black ball and a white ball are drawn in the first draw, respectively. Then,

$$\begin{aligned}
\mathbb{P}(A_{\text{black}}) &= \frac{n}{n+b}, \\
\mathbb{P}(A_{\text{white}}) &= \frac{b}{n+b}.
\end{aligned}$$

Next, let B_{black} be the event that a black ball is drawn in the second draw. Conditioning on the first draw, we obtain:

$$\begin{aligned}
\mathbb{P}_{A_{\text{black}}}(B_{\text{black}}) &= \frac{n+d}{n+b+d}, \\
\mathbb{P}_{A_{\text{white}}}(B_{\text{black}}) &= \frac{n}{n+b+d}.
\end{aligned}$$

Since $A_{\text{black}} \cup A_{\text{white}} = \Omega$ and $\mathbb{P}(A_{\text{black}} \cup A_{\text{white}}) = 1$, by the law of total probability we have:

$$\begin{aligned}
\mathbb{P}(B_{\text{black}}) &= \mathbb{P}(A_{\text{black}} \cap B_{\text{black}}) + \mathbb{P}(A_{\text{white}} \cap B_{\text{black}}) \\
&= \mathbb{P}(A_{\text{black}})\, \mathbb{P}_{A_{\text{black}}}(B_{\text{black}}) + \mathbb{P}(A_{\text{white}})\, \mathbb{P}_{A_{\text{white}}}(B_{\text{black}}) \\
&= \frac{n}{n+b} \cdot \frac{n+d}{n+b+d} + \frac{b}{n+b} \cdot \frac{n}{n+b+d} \\
&= \frac{n(n+d)+nb}{(n+b)(n+b+d)}.
\end{aligned}$$

Finally, applying Bayes' theorem, the conditional probability that the first drawn ball was black given that the second draw was black is:

$$\begin{aligned}
\mathbb{P}_{B_{\text{black}}}(A_{\text{black}}) &= \frac{\mathbb{P}_{A_{\text{black}}}(B_{\text{black}})\, \mathbb{P}(A_{\text{black}})}{\mathbb{P}(B_{\text{black}})} \\
&= \frac{\frac{n+d}{n+b+d} \cdot \frac{n}{n+b}}{\frac{n(n+d)+nb}{(n+b)(n+b+d)}} \\
&= \frac{n+d}{n+d+b}.
\end{aligned}$$

Answer

$$\boxed{\mathbb{P}_{B_{\text{black}}}(A_{\text{black}}) = \frac{n+d}{n+d+b}}$$