Question 3: Conditional Prize Award in a Sequence of Matches

Problem Statement

To obtain a prize, you need to win *at least two consecutive matches* out of the three you play, alternating between matches against your father and your coach. You win 75% of the matches against your father and 40% of the matches against your coach.

Preliminaries

Let win and lose denote the outcomes of a match, and let F and C denote the events of winning against the father and the coach, respectively. Define the probability space as

$$\begin{aligned}
&\Omega = \{\text{win, lose}\},\\
&\mathcal{F} = \mathcal{P}(\Omega),\\
&\mathbb{P}: \quad \mathbb{P}(F) = 75\%, \quad \mathbb{P}(C) = 40\%.
\end{aligned}$$

Let W be the event of obtaining a prize (i.e., winning at least two consecutive matches) and denote by A, B, and C the outcomes of the first, second, and third matches respectively. To be specific, we use subscripts f and c to indicate matches played against the father and the coach. For example, A_f means winning the first match against the father, while A_c means winning the first match against the coach.

Assuming that the match outcomes are stochastically independent, we have:

$$\begin{aligned}
\mathbb{P}(A_f) = \mathbb{P}(B_f) = \mathbb{P}(C_f) = \mathbb{P}(F) = 75\%,\\
\mathbb{P}(A_c) = \mathbb{P}(B_c) = \mathbb{P}(C_c) = \mathbb{P}(C) = 40\%.
\end{aligned}$$

Part 1: Comparing Two Playing Sequences

We consider the following two sequences of matches:

- **Father-Coach-Father** (denoted by W_{FCF})
- **Coach-Father-Coach** (denoted by W_{CFC})

Sequence 1: Father-Coach-Father

In this sequence, a prize is obtained if at least two consecutive wins occur. One can show that the event of winning a prize in the Father-Coach-Father sequence is given by:

$$\begin{aligned}\mathbb{P}(W_{FCF}) &= \mathbb{P}\Big((A_f \cap B_c \cap C_f) \cup (A_f^c \cap B_c \cap C_f) \cup (A_f \cap B_c \cap C_f^c)\Big) \\ &= 2\,\mathbb{P}(F)\,\mathbb{P}(C) - \mathbb{P}(F)^2\,\mathbb{P}(C).\end{aligned}$$

Sequence 2: Coach-Father-Coach

Similarly, for the Coach-Father-Coach sequence, the probability of obtaining a prize is:

$$\begin{aligned}\mathbb{P}(W_{CFC}) &= \mathbb{P}\Big((A_c \cap B_f \cap C_c) \cup (A_c^c \cap B_f \cap C_c) \cup (A_c \cap B_f \cap C_c^c)\Big) \\ &= 2\,\mathbb{P}(C)\,\mathbb{P}(F) - \mathbb{P}(C)^2\,\mathbb{P}(F).\end{aligned}$$

Since $\mathbb{P}(F) = 75\%$ and $\mathbb{P}(C) = 40\%$, we have

$$\mathbb{P}(F) > \mathbb{P}(C),$$

which implies

$$\mathbb{P}(W_{FCF}) - \mathbb{P}(W_{CFC}) = \mathbb{P}(F)\,\mathbb{P}(C)\Big(\mathbb{P}(C) - \mathbb{P}(F)\Big) < 0.$$

Thus,

$$\boxed{\mathbb{P}(W_{CFC}) > \mathbb{P}(W_{FCF})}.$$

Answer for Part 1: The sequence *Coach-Father-Coach* is preferable.

Part 2: Including Additional Outcomes

Let W' be the event of obtaining a prize when additional outcomes are considered.

Sequence 1: Modified Father-Coach-Father (W'_{FCF})

For the Father-Coach-Father sequence, we now have:

$$\begin{aligned}\mathbb{P}(W'_{FCF}) &= \mathbb{P}\Big(W_{FCF} \cup \big(A_f \cap B_c^c \cap C_f\big)\Big) \\ &= 2\,\mathbb{P}(F)\,\mathbb{P}(C) + \mathbb{P}(F)^2\Big(1 - 2\,\mathbb{P}(C)\Big) \\ &\approx 71.25\%.\end{aligned}$$

Sequence 2: Modified Coach-Father-Coach (W'_{CFC})

Similarly, for the Coach-Father-Coach sequence, we have:

$$\begin{aligned}\mathbb{P}(W'_{CFC}) &= \mathbb{P}\Big(W_{CFC} \cup \big(A_c \cap B_f^c \cap C_c\big)\Big) \\ &= 2\,\mathbb{P}(C)\,\mathbb{P}(F) + \mathbb{P}(C)^2\Big(1 - 2\,\mathbb{P}(F)\Big) \\ &\approx 52\%.\end{aligned}$$

Since

$$\mathbb{P}(W'_{FCF}) \approx 71.25\% > 52\% \approx \mathbb{P}(W'_{CFC}),$$

Answer for Part 2: The strategy should be changed to *Father-Coach-Father*.