Question 4: Analysis of an Alternating Coin Toss Game

Problem Statement

Two players flip a fair coin alternately. The game ends when one of them gets heads.

Part 1: Probability of a Specific Sequence of Outcomes

Let head and tail denote the outcomes *head* and *tail* respectively. We define the probability space by

$$\begin{aligned}
&\Omega = \{\text{head, tail}\},\\
&\mathcal{F} = \mathcal{P}(\Omega),\\
&\mathbb{P}: \quad \mathbb{P}(\{\text{head}\}) = \mathbb{P}(\{\text{tail}\}) = \frac{1}{2}.
\end{aligned}$$

Let $T_i \subset \mathcal{F}$ be the event that the i-th flip yields a tail:

$$T_i = \{\text{tail}\}, \quad i = 1, 2, \dots, n-1,$$

and let $H_n \subset \mathcal{F}$ be the event that the n-th flip yields a head:

$$H_n = \{\text{head}\}.$$

Since the coin tosses are independent, the probability that the first $n-1$ flips are tails and the n-th flip is heads is

$$\begin{aligned}
\mathbb{P}\Big(T_1 \cap T_2 \cap \cdots \cap T_{n-1} \cap H_n\Big) &= \mathbb{P}(T_1)\,\mathbb{P}(T_2)\cdots\mathbb{P}(T_{n-1})\,\mathbb{P}(H_n)\\
&= \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2}\\
&= \frac{1}{2^n}.
\end{aligned}$$

Answer for Part 1:

$$\boxed{\mathbb{P}(T_1 \cap T_2 \cap \cdots \cap T_{n-1} \cap H_n) = \frac{1}{2^n}}.$$

Part 2: Probability that the First Player Wins

Assume that the players alternate flips, with the first player flipping on the odd-numbered turns and the second player on the even-numbered turns. Let us denote:

- $T'_{i'} = \{\text{tail}\}$ for the i'-th flip,
- $H'_{2k+1} = \{\text{head}\}$ corresponding to the first player's flip on turn $2k+1$ (with $k \geq 0$, $k \in \mathbb{N}$).

Let F be the event that the first player wins. This happens if the first flip is a head, or if the first head appears on any odd-numbered flip (after an even number of tails), i.e.,

$$\begin{aligned}
\mathbb{P}(F) &= \mathbb{P}(H'_1) + \sum_{k=1}^{\infty} \mathbb{P}\Big(T'_1 \cap T'_2 \cap \cdots \cap T'_{2k} \cap H'_{2k+1}\Big) \\
&= \mathbb{P}(H'_1) + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k+1}.
\end{aligned}$$

Noting that $\mathbb{P}(H'_1) = \frac{1}{2}$, we can rewrite the sum as:

$$\mathbb{P}(F) = \frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k+1} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2k+1}.$$

Factor out $\frac{1}{2}$:

$$\mathbb{P}(F) = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k}.$$

Recognizing the sum as a geometric series with common ratio $r = \frac{1}{4}$, where

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r},$$

we obtain:

$$\mathbb{P}(F) = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{2} \cdot \frac{1}{\frac{3}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

Answer for Part 2:

$$\boxed{\mathbb{P}(F) = \frac{2}{3}}.$$