Question 4

Consider a routine screening test for a disease. Suppose the frequency of the disease in the population (base rate) is 0.5%. The test is fairly accurate with a 5% false positive rate and a 10% false negative rate. You take the test and it comes back positive. What is the probability that you have the disease?

Solution

Let *positive* and *negative* denote the results of a test. Define the probability space as

$$\begin{aligned}
&\Omega = \{\text{positive, negative}\} \\
&\mathcal{F} = \mathcal{P}(\Omega) \\
&\mathbb{P}: \ \text{the probability measure on } \mathcal{F}
\end{aligned}$$

Let B be the event that having the disease in the population (base rate)

$$\mathbb{P}(B) = 0.5\%$$

Then we have the false positive rate ($\mathbb{P}_{B^c}(\{\text{positive}\})$) and the false negative rate ($\mathbb{P}_B(\{\text{negative}\})$)

$$\begin{aligned}
\mathbb{P}_{B^c}(\{\text{positive}\}) &= 5\% \\
\mathbb{P}_B(\{\text{negative}\}) &= 10\%
\end{aligned}$$

Then we have the true negative rate ($\mathbb{P}_{B^c}(\{\text{negative}\})$) and the true positive rate ($\mathbb{P}_B(\{\text{positive}\})$)

$$\begin{aligned}
\mathbb{P}_{B^c}(\{\text{negative}\}) = 1 - \mathbb{P}_{B^c}(\{\text{positive}\}) &\quad = 95\% \\
\mathbb{P}_B(\{\text{positive}\}) = 1 - \mathbb{P}_B(\{\text{negative}\}) &\quad = 90\%
\end{aligned}$$

Then by applying the law of total probability, we have

$$\mathbb{P}(\{\text{positive}\}) = \mathbb{P}(B)\mathbb{P}_B(\{\text{positive}\}) + \mathbb{P}(B^c)\mathbb{P}_{B^c}(\{\text{positive}\}) = 5.425\%$$

By applying the Bayes' Theorem, we have

$$\mathbb{P}_{\{\text{positive}\}}(B) = \frac{\mathbb{P}_B(\{\text{positive}\})\mathbb{P}(B)}{\mathbb{P}(\{\text{positive}\})} = \frac{18}{217} \approx 8.295\%$$

Answer

$$\boxed{\mathbb{P}_{\{\text{positive}\}}(B) = \frac{18}{217} \approx 8.295\%}$$