Question 1

Consider a binary communication channel, with input X having a Bernoulli distribution with parameter $p = 0.9$. The common error probability is $\epsilon = 0.05$ (i.e., the probability that the received character differs from the input character is ϵ). Let Y denote the output character.

Part 1: Show that Y is a Bernoulli distribution with parameter q.

Solution

We know that X has a Bernoulli distribution in sample space $\Omega = \{0, 1\}$ (let 0 denotes *not received* and 1 denotes *received*) with parameter $p = 0.9$, then easy to see that

$$\begin{aligned}
\mathbb{P}(Y = 0) &= \mathbb{P}(X = 0)\mathbb{P}_{X=0}(Y = 0) + \mathbb{P}(X = 1)\mathbb{P}_{X=1}(Y = 0) \\
&= (1 - p)(1 - \epsilon) + p\epsilon \\
&= 2p\epsilon - p - \epsilon + 1
\end{aligned}$$

Similiarly, we have

$$\begin{aligned}
\mathbb{P}(Y = 1) &= \mathbb{P}(X = 0)\mathbb{P}_{X=0}(Y = 1) + \mathbb{P}(X = 1)\mathbb{P}_{X=1}(Y = 1) \\
&= (1 - p)\epsilon + p(1 - \epsilon) \\
&= -2p\epsilon + p + \epsilon
\end{aligned}$$

Then we have

$$0 < \mathbb{P}(Y = 0), \mathbb{P}(Y = 1) < 1, \quad \text{and} \quad \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) = 1$$

Then we can say that Y is a Bernouli distribution in sample space $\Omega = \{0, 1\}$ with parameter

$$\begin{aligned}
q &= \mathbb{P}(Y = 1) \\
&= -2p\epsilon + p + \epsilon
\end{aligned}$$

Answer

$$\boxed{\text{See above.}}$$

Part 2: Determine q.

Solution

With the explanation and formula above, easy to see that

$$q = -2p\epsilon + p + \epsilon = 0.86$$

Answer

$$\boxed{q = 0.86}$$

Part 3: Compute the joint probability distribution function of (X, Y).

Solution

Easy to compute the joint distribution function of (X, Y) as

$$f(x,y) = \begin{cases} \mathbb{P}(X=0,Y=0) = \mathbb{P}(X=0)\mathbb{P}_{X=0}(Y=0) = & (1-p)1-\epsilon = 0.095, & x=0, & y=0 \\ \mathbb{P}(X=0,Y=1) = \mathbb{P}(X=0)\mathbb{P}_{X=0}(Y=1) = & (1-p)\epsilon = 0.005, & x=0, & y=1 \\ \mathbb{P}(X=1,Y=0) = \mathbb{P}(X=1)\mathbb{P}_{X=1}(Y=0) = & p\epsilon = 0.045, & x=1, & y=0 \\ \mathbb{P}(X=1,Y=1) = \mathbb{P}(X=1)\mathbb{P}_{X=1}(Y=1) = & p(1-\epsilon) = 0.855, & x=1, & y=1 \end{cases}$$

Answer

$$\boxed{f(x,y) = \begin{cases} 0.095, & x=0, & y=0 \\ 0.005, & x=0, & y=1 \\ 0.045, & x=1, & y=0 \\ 0.855, & x=1, & y=1 \end{cases}}$$