Question 4

We are given a bowl that contains 6 balls, numbered from 1 to 6. We extract two balls and denote X and Y the numbers on the ball obtained at the first (second) extraction, and $W = \max(X, Y)$ the maximum value obtained. In all scenarios, describe the probability distribution of W.

Part 1: In the first scenario, assume that the extractions are made with replacement.

Solution

By applying the principle of symmetry, easy to define the probability space as

$$\begin{aligned}
&\Omega = \{1, 2, \dots, 6\} \\
&\mathcal{F} = \mathcal{P}(\Omega) \\
&\mathbb{P}: \ \mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\}) = \frac{1}{6}
\end{aligned}$$

Then as given, we have

$$\begin{aligned}
X, Y:&\ \ \Omega \to R \\
W &= \max(X, Y) \\
R &= \{1, 2, \dots, 6\}
\end{aligned}$$

Then we have the probability distribution of W in scenario #1 (denoted as W_1 and p_1)

$$p_1(w_1) = \begin{cases}
\mathbb{P}(W_1 = 1) = \mathbb{P}(\{(1,1)\}) = \frac{1}{6^2} = \frac{1}{36}, & w_1 = 1 \\
\mathbb{P}(W_1 = 2) = \mathbb{P}(\{(1,2), (2,1), (2,2)\}) = \frac{3}{36}, & w_1 = 2 \\
\mathbb{P}(W_1 = 3) = \mathbb{P}(\{(1,3), (2,3), \dots, (3,3)\}) = \frac{5}{36}, & w_1 = 3 \\
\mathbb{P}(W_1 = 4) = \mathbb{P}(\{(1,4), (2,4), \dots, (4,4)\}) = \frac{7}{36}, & w_1 = 4 \\
\mathbb{P}(W_1 = 5) = \mathbb{P}(\{(1,5), (2,5), \dots, (5,5)\}) = \frac{9}{36}, & w_1 = 5 \\
\mathbb{P}(W_1 = 6) = \mathbb{P}(\{(1,6), (2,6), \dots, (6,6)\}) = \frac{11}{36}, & w_1 = 6
\end{cases}$$

Or a general formula without cases

$$p_1(w_1) = \frac{2w_1 - 1}{6^2} = \frac{2w_1 - 1}{36}, \quad w_1 \in \{1, 2, \dots, 6\}$$

Answer

$$\boxed{p_1(w_1) = \begin{cases} \frac{1}{36}, & w_1 = 1 \\ \frac{3}{36}, & w_1 = 2 \\ \frac{5}{36}, & w_1 = 3 \\ \frac{7}{36}, & w_1 = 4 \\ \frac{9}{36}, & w_1 = 5 \\ \frac{11}{36}, & w_1 = 6 \end{cases}}$$

Part 2: In the second scenario, assume that the extractions are performed without replacement.

Solution

Easy to see that

$$p_2(w_2) = \begin{cases} \mathbb{P}(W_2 = 2) = \mathbb{P}(\{(1,2),(2,1)\}) & = \frac{2}{6\times 5} = \frac{1}{15}, \quad w_2 = 2 \\ \mathbb{P}(W_2 = 3) = \mathbb{P}(\{(1,3),(2,3),\ldots,(3,2)\}) & = \frac{2}{15}, \quad w_2 = 3 \\ \mathbb{P}(W_2 = 4) = \mathbb{P}(\{(1,4),(2,4),\ldots,(4,3)\}) & = \frac{3}{15}, \quad w_2 = 4 \\ \mathbb{P}(W_2 = 5) = \mathbb{P}(\{(1,5),(2,5),\ldots,(5,4)\}) & = \frac{4}{15}, \quad w_2 = 5 \\ \mathbb{P}(W_2 = 6) = \mathbb{P}(\{(1,6),(2,6),\ldots,(6,5)\}) & = \frac{5}{15}, \quad w_2 = 6 \end{cases}$$

Or a general formula without cases

$$p_2(w_2) = \frac{2w_2 - 2}{6 \times 5} = \frac{w_2 - 1}{15}, \quad w_2 \in \{2, 3, \ldots, 6\}$$

Answer

$$p_2(w_2) = \begin{cases} \frac{1}{15}, & w_2 = 2 \\ \frac{2}{15}, & w_2 = 3 \\ \frac{3}{15}, & w_2 = 4 \\ \frac{4}{15}, & w_2 = 5 \\ \frac{5}{15}, & w_2 = 6 \end{cases}$$

Part 3: In the third scenario, assume that after the first extraction, we replace the ball in the urn, together with another one with the same number.

Solution

Easy to see that

$$p_3(w_3) = \begin{cases} \mathbb{P}(W_3 = 1) = \mathbb{P}(\{(1,1)\}) & = \frac{1}{6} \times \frac{2}{7} = \frac{1}{21}, & w_3 = 1 \\ \mathbb{P}(W_3 = 2) = \mathbb{P}(\{(1,2),(2,1),(2,2)\}) & = 2\left(\frac{1}{6} \times \frac{1}{7}\right) + \frac{2}{42} = \frac{2}{21}, & w_3 = 2 \\ \mathbb{P}(W_3 = 3) = \mathbb{P}(\{(1,3),(2,3),\ldots,(3,3)\}) & = 4\left(\frac{1}{42}\right) + \frac{2}{42} = \frac{3}{21}, & w_3 = 3 \\ \mathbb{P}(W_3 = 4) = \mathbb{P}(\{(1,4),(2,4),\ldots,(4,4)\}) & = 6\left(\frac{1}{42}\right) + \frac{2}{42} = \frac{4}{21}, & w_3 = 4 \\ \mathbb{P}(W_3 = 5) = \mathbb{P}(\{(1,5),(2,5),\ldots,(5,5)\}) & = 8\left(\frac{1}{42}\right) + \frac{2}{42} = \frac{5}{21}, & w_3 = 5 \\ \mathbb{P}(W_3 = 6) = \mathbb{P}(\{(1,6),(2,6),\ldots,(6,6)\}) & = 10\left(\frac{1}{42}\right) + \frac{2}{42} = \frac{6}{21}, & w_3 = 6 \end{cases}$$

Or a general formula without cases

$$p_3(w_3) = \frac{w_3}{21}, \quad w_3 \in \{1, 2, \ldots, 6\}$$

Answer

$$\boxed{p_3(w_3) = \begin{cases} \dfrac{1}{21}, & w_3 = 1 \\ \dfrac{2}{21}, & w_3 = 2 \\ \dfrac{3}{21}, & w_3 = 3 \\ \dfrac{4}{21}, & w_3 = 4 \\ \dfrac{5}{21}, & w_3 = 5 \\ \dfrac{6}{21}, & w_3 = 6 \end{cases}}$$