Question 1

Hugo is trying to complete his card collection and needs just one last card. He can obtain cards from biscuit packs, and each pack has a 0.2 probability of containing the card he needs. Hugo has enough money to buy at most 4 packs, but he will stop buying as soon as he finds the missing card. Let X be the number of packs Hugo buys.

Part 1: What is the probability distribution of X?

Solution

As given, easy to know that X has a geometric distribution with parameter p, that is

$$X \sim \text{Geo}(p), \quad p = 0.2, \quad X \in \{1, 2, 3, 4\}$$

We know that

$$\mathbb{P}(X = k) = (1-p)^{k-1}p, \quad k \in X$$

Then we have

$$\mathbb{P}(X = k) = \begin{cases} (1-0.2)^{1-1}0.2 = \dfrac{1}{5} & = 0.2, \quad k = 1 \\ (1-0.2)^{2-1}0.2 = \dfrac{4}{25} & = 0.16, \quad k = 2 \\ (1-0.2)^{3-1}0.2 = \dfrac{16}{125} & = 0.128, \quad k = 3 \\ 1 - \displaystyle\sum_{i=1}^{3} \mathbb{P}(X = i) = \dfrac{64}{125} & = 0.512, \quad k = 4 \end{cases}$$

Answer

$$\boxed{\mathbb{P}(X = k) = \begin{cases} \dfrac{1}{5} = 0.2, & k = 1 \\ \dfrac{4}{25} = 0.16, & k = 2 \\ \dfrac{16}{125} = 0.128, & k = 3 \\ \dfrac{64}{125} = 0.512, & k = 4 \end{cases}}$$

Part 2: What is the probability that Hugo successfully completes his collection?

Solution

Let event C denotes that Hugo completes his collection and let C_4 denotes that Hugo completes his collection when opening the fourth pack, easy to know that

$$C = C_4 \cup \bigcup_{k=1}^{3} (X = k), \quad \mathbb{P}(C_4) = (1-p)^{4-1}p$$

Then we have

$$\begin{aligned}
\mathbb{P}(C) &= \mathbb{P}(C_4) + \sum_{k=1}^{3} \mathbb{P}(X = k) \\
&= (1-0.2)^3 0.2 + \frac{1}{5} + \frac{4}{25} + \frac{16}{125} \\
&= \frac{369}{625} \\
&= 0.590\,4
\end{aligned}$$

Answer

$$\boxed{\mathbb{P}(C) = \frac{369}{625} = 0.590\,4}$$

Part 3: Given that Hugo completes the collection, what is the probability that he only buys one pack?

Solution

By applying the Bayes' Theorem, we know that

$$\begin{aligned}
\mathbb{P}_C(X = 1) &= \frac{\mathbb{P}_{X=1}(C)\mathbb{P}(X = 1)}{\mathbb{P}(C)} \\
&= \frac{1 \times \frac{1}{5}}{\frac{369}{625}} \\
&= \frac{125}{369} \\
&\approx 0.339
\end{aligned}$$

Answer

$$\boxed{\mathbb{P}_C(X = 1) = \frac{125}{369} \approx 0.339}$$