# Question 3

Alice proposes to Bob the following bet. Alice tosses a fair coin n times, and computes the number of heads $X$. Bob tosses the coin $n+1$ times, and obtain $Y$ heads. Bob wins the bet if $Y > X$.

**Part 1:** Is the bet fair?

## Solution

As given, we can define the Bernoulli distribution with sample space $\Omega = \{\text{head}, \text{tail}\}$ (*head* denotes *drop headed* and *tail* denotes *drop tailed*) and with parameter $p = 0.5$ (by applying the principle of symmetry), then we can define the Binomial distribution of both $X$ and $Y$ as

$$\begin{aligned} X \sim& \text{Bin}(n, p) \\ Y \sim& \text{Bin}(n+1, p) \end{aligned}$$

Let as define $Z = Y - X$, then we have

$$Z \sim \text{Bin}(1, p)$$

Then we have the distribution of $Z$

$$p(z) = \begin{cases} 0.5, & z = 0 \\ 0.5, & z = 1 \end{cases}$$

That is

$$\begin{aligned} \mathbb{P}(Y > X) &= \mathbb{P}(Z > 0) \\ &= \frac{p(1)}{p(0) + p(1)} \\ &= \frac{1}{2} \end{aligned}$$

## Answer

$$\boxed{\text{Fair.}}$$

**Part 2:** Compute the answer for a general coin.

## Solution

Here we let $p_g$ denote a random variable in $(0, 1)$, then we have the $Z$ in general case

$$Z_g \sim \text{Bin}(1, p_g)$$

Then we have the distribution of $Z_g$

$$p_g(z) = \begin{cases} 1 - p_g, & z = 0 \\ p_g, & z = 1 \end{cases}$$

By denoting $X$ and $Y$ in general case as $X_g$ and $Y_g$, we have

$$\begin{aligned} \mathbb{P}(Y_g > X_g) &= \mathbb{P}(Z_g > 0) \\ &= \frac{p_g(1)}{p_g(0) + p_g(1)} \\ &= p_g \end{aligned}$$

That is

$$\text{Bet is} \begin{cases} \text{Unfair, Alice tends to win more,} & p_g < 0.5 \\ \text{Fair,} & p_g = 0.5 \\ \text{Unfair, Bob tends to win more,} & p_g > 0.5 \end{cases}$$

## Answer

$$\boxed{\text{Bet is} \begin{cases} \text{Unfair, Alice tends to win more,} & p_g < 0.5 \\ \text{Fair,} & p_g = 0.5 \\ \text{Unfair, Bob tends to win more,} & p_g > 0.5 \end{cases}}$$