Question 3

Let X denote the number of heads when tossing two fair coins.

Part 1: Construct the probability distribution of X. Compute the average and the standard deviation of X.

Solution

By applying the principle of symmetry, easy to define the probability space as

$$\begin{aligned}
&\Omega = \{\text{head}, \text{tail}\} \\
&\mathcal{F} = \mathcal{P}(\Omega) \\
&\mathbb{P}: \ \ \mathbb{P}(\{(\text{head}, \text{head})\}) = \mathbb{P}(\{(\text{head}, \text{tail})\}) = \mathbb{P}(\{(\text{tail}, \text{head})\}) = \mathbb{P}(\{(\text{tail}, \text{tail})\}) = \frac{1}{4}
\end{aligned}$$

Then we have the *pdf* of X

$$p_X(x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases}$$

Then we have the average of X

$$\begin{aligned}
\mu_X &= 0p_X(0) + 1p_X(1) + 2p_X(2) \\
&= 0 + \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}$$

As well as the expectation of X^2

$$\begin{aligned}
\mathbb{E}\left[X^2\right] &= 0^2p_X(0) + 1^2p_X(1) + 2^2p_X(2) \\
&= 0 + \frac{1}{2} + 1 \\
&= \frac{3}{2}
\end{aligned}$$

Then we have the standard deviation of X

$$\begin{aligned}\sigma_X &= \sqrt{\mathbb{E}\left[X^2\right] - {\mu_X}^2} \\ &= \sqrt{\frac{1}{2}} \\ &\approx 0.707\end{aligned}$$

Answer

$$\boxed{\begin{aligned} p_X(x) &= \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases} \\ \mu_X &= 1 \\ \sigma_X &= \sqrt{\frac{1}{2}} \approx 0.707 \end{aligned}}$$

Part 2: Now suppose you toss a fair coin repeatedly until you get a tail. Let Y be the number of heads obtained before the first tail. Construct the probability distribution of Y. Compute the average and the standard deviation of Y and compare them with those of X.

Solution

Easy to notice that Y follows a geometric distribution with parameter $\frac{1}{2}$ (by applying the principle of symmetry), then we have the *pdf* of Y

$$p_Y(y) = \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right) = \frac{1}{2^{y+1}}, \quad y \in \{0, 1, \dots\}$$

Then we have the average of Y

$$\begin{aligned}\mu_Y &= \sum_{y=0}^{\infty} \frac{y}{2^{y+1}} \\ &= \frac{1}{2} \sum_{y=0}^{\infty} y \left(\frac{1}{2}\right)^y \\ &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2}\right) \\ &= 1\end{aligned}$$

As well as the expectation of Y^2

$$
\begin{aligned}
\mathbb{E}\left[Y^2\right] &= \sum_{y=0}^{\infty} \frac{y^2}{2^{y+1}} \\
&= \frac{1}{2} \sum_{y=0}^{\infty} y^2 \left(\frac{1}{2}\right)^y \\
&= \frac{1}{2} \left(\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{\left(1-\frac{1}{2}\right)^3} \right) \\
&= 3
\end{aligned}
$$

Then we have the standard deviation of Y

$$
\begin{aligned}
\sigma_Y &= \sqrt{\mathbb{E}\left[Y^2\right] - {\mu_Y}^2} \\
&= \sqrt{2} \\
&\approx 1.414
\end{aligned}
$$

Comparison as

$$\mu_X = 1 = \mu_Y$$

$$\sigma_X = \sqrt{\frac{1}{2}} \approx 0.707 < 1.414 \approx \sqrt{2} = \sigma_Y$$

Answer

$$
\boxed{
\begin{aligned}
p_Y(y) &= \frac{1}{2^{y+1}}, \quad y \in \{0, 1, \dots\} \\
\mu_Y &= 1 \\
\sigma_Y &= \sqrt{2} \approx 1.414 \\
\mu_Y &= \mu_X \\
\sigma_Y &> \sigma_X
\end{aligned}
}
$$